Section 3.1 – Solving Systems by Graphing

Objective: To solve linear systems graphically.

Systems of Equations

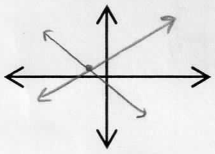
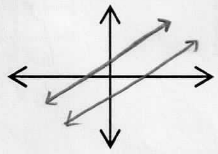
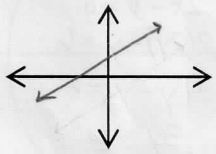
A system of equations is 2 or more equations with the same variables. To solve a system of equations, find the ordered pair that satisfies all the equations. There are four methods to solve a system: Graphing, Substitution, Elimination, and Matrices.

Classification of Systems

We classify the solutions of a system by two different characteristics.

1. Consistent or Inconsistent
   - A system is Consistent if it has at least one solution
   - A system is Inconsistent if it has no solutions
2. Dependent or Independent
   - A system is Independent if it has exactly one solution
   - A system is Dependent if it has an infinite number of solutions. (infinitely many)

There are three possibilities for the number of solutions of a 2x2 system of linear equations:

Possibility:	1. Intersecting Lines	2. Parallel Lines	3. Same Line
What does it look like graphically?			
How many solutions?	One solution	No solution	Infinitely Many Solutions
Classification?	Consistent Independent	Inconsistent	Consistent Dependent
What happens algebraically?	When you solve, you get one x-value & one y-value	When you solve you get a false statement	When you solve you get a true statement.

$2x+5=2x-3$
$5 \neq -3$
False

$2x+5=2x+5$
$5=5$
True

Classifying a System without graphing:

Without graphing, is the system $\begin{cases} -3x+y=4 \\ x-\frac{1}{3}y=1 \end{cases}$ consistent, independent; consistent, dependent; or inconsistent?

$-3x+y=4$ $\quad$ $\boxed{y=3x+4}$

$x-\frac{1}{3}y=1$ $\quad$ $-3(-\frac{1}{3}y=-x+1)$ $\quad$ $\boxed{y=3x-3}$

- Write each equation in slope intercept form:
- If the slopes are the same and the y-intercepts are the same then it is consistent dependent (same line)
- If the slopes are the same and the y-intercepts are different then it is inconsistent.
- If the slopes are different then it is consistent, independent.

∴ this system is Inconsistent

Solve the system of equations by graphing. *Mentally check your answers.* Then describe the type of lines.

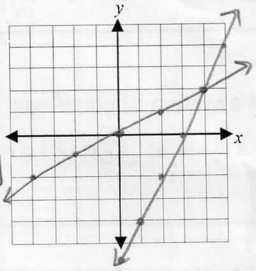
$y = mx + b$

1. $x = 2y$ $\quad \frac{x}{2} = \frac{2y}{2}$ $\quad \boxed{y = \frac{1}{2}x}$

$2x - y = 6$

$-y = -2x + 6$

$\boxed{y = 2x - 6}$

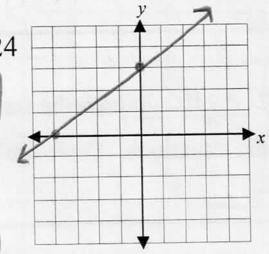


2. $6x - 8y = -24$

$\boxed{y = \frac{3}{4}x + 3}$

$\frac{-8y}{-8} = \frac{-6x - 24}{-8}$

$\boxed{y = \frac{3}{4}x + 3}$



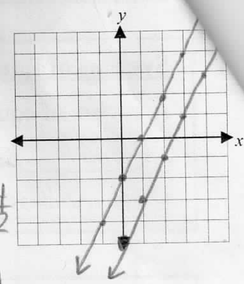
3. $y + 5 = 2x$

$4x - 2y = 4$

$\boxed{y = 2x - 5}$

$\frac{-2y}{-2} = \frac{-4x}{-2} + \frac{4}{-2}$

$\boxed{y = 2x - 2}$



Solution (4,2) | Solution Infinitely Many | Solution $\emptyset$ no solution (parallel)

Classification Consistent Independent | Classification Consistent Dependent | Classification Inconsistent

Section 3.2 – Solving Systems Algebraically

Objective: To solve systems algebraically

- Solve the system of equations using the Substitution method.

1. $x = 2y$

$2x - y = 6$

$2(2y) - y = 6$

$4y - y = 6$

$\frac{3y}{3} = \frac{6}{3}$

$\boxed{y = 2}$ Plug into other original

$x = 2(2)$

$\boxed{x = 4}$

Solution: (4,2) Consistent Independent

2. $6x - 8y = -24$

$y = \frac{3}{4}x + 3$

$6x - 8(\frac{3}{4}x + 3) = -24$

$6x - 6x - 24 = -24$

$+24 \quad +24$

$0 = 0$ True ✓

Infinitely many solutions Consistent Dependent

3. $y + 5 = 2x \rightarrow y = 2x - 5$

$4x - 2y = 4$

$4x - 2(2x - 5) = 4$

$4x - 4x + 10 = 4$

$10 \neq 4$ False

No solution Inconsistent

4. $x - y - z = -4$

$5x + 2y - 3z = \cancel{-14}\ 7$

$\frac{6z}{6} = \frac{-24}{6}$

$\boxed{z = -4}$

$x - y - -4 = -4$

$x - y + 4 = -4$

$x - y = -8$

$x = y - 8$

$5x + 2y - 3(-4) = 7$

$5x + 2y + 12 = -5$

$5x + 2y = -5$

$5(y - 8) + 2y = -5$

$5y - 40 + 2y = -5$

$\frac{7y}{7} = \frac{35}{7}$

$\boxed{y = 5}$

$x - (5) - (-4) = -4$

$x - 5 + 4 = -4$

$x - 1 = -4$

$\boxed{x = -3}$

- Solve the system of equations using the Elimination method. *Be sure to check your answers.*

1. $3x - 2y = 2$

$2(2x + y = 13) \rightarrow 2(4) + y = 13$

$4x + 2y = 26 \qquad 8 + y = 13$

$3x - 2y = 2 \qquad -8 \quad -8$

$\frac{7x}{7} = \frac{28}{7} \qquad \boxed{y = 5}$

$\boxed{x = 4}$

(4,5) Consistent Dependent

2. $5y = 4 - 3x \rightarrow 5(\frac{1}{2}) = 4 - 3x$

$-3(5x + 7y = 6)$

$5(3x + 5y = 4)$

$-15x - 21y = -18$

$15x + 25y = 20$

$\frac{4y}{4} = \frac{2}{4}$

$\boxed{y = \frac{1}{2}}$

Consistent Dependent

$\frac{5}{2} = 4 - 3x$

$\frac{5}{2} - \frac{4}{1} = -3x$

$\frac{5}{2} - \frac{8}{2} = -3x$

$-\frac{1}{3} \cdot -\frac{3}{2} = \frac{-3x}{-3}$

$\boxed{\frac{1}{2} = x}$

3. $2x + 6y = 10 - 2x$

$6x = 10 - 9y$

$-3(4x + 6y = 10)$

$2(6x + 9y = 10)$

$-12x - 18y = -30$

$12x + 18y = 20$

$0 = -10$

no solution inconsistent

Section 3.3 – System of Linear Inequalities Objective: To solve a system of inequalities

System of Linear Inequalities:

To solve a system of inequalities, we need to find the ordered pairs that satisfy all of the equations in the system. The solution set is represented by the intersection of shaded regions of the graphs of the inequalities.

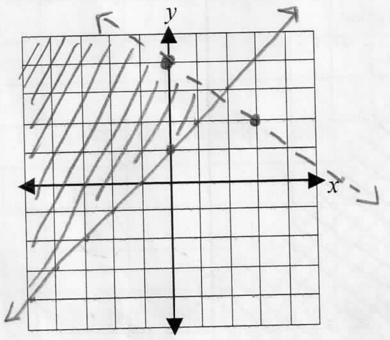
Graph the System **of** Inequalities.

For each inequality:

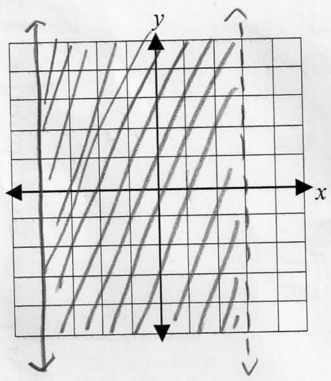
1. Graph the boundary line.
2. Determine whether the boundary line is solid or dashed.
3. lightly the half-plane.

Darken the overlap of the shading for all of the inequalities.

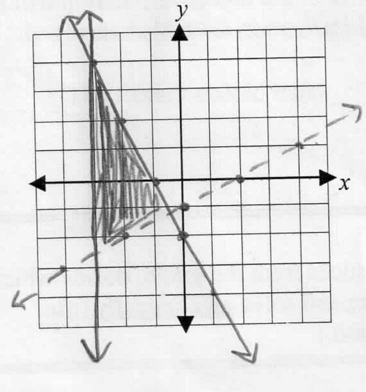
1. $y < -\frac{2}{3}x + 4$
   $y \geq x + 1$



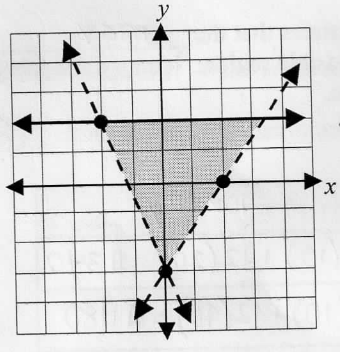
2. $-4 \leq x < 3$



3. $2x + y \leq -2$ $y \leq -2x - 2$
   $y > \frac{1}{2}x - 1$
   $x \geq -3$



4. Write the system of inequalities from the graph.



$$\begin{cases} y \leq 2 \\ y \geq \frac{3}{2}x - 3 \\ y \geq \frac{-5x}{2} - 3 \end{cases}$$

Section 3.4 – Linear Programming

Objective: To solve real world problems using linear programming.

Linear Programming is the process of Finding Maximum and minimum values of a function for a region defined by inequalities. The inequalities are called Constraints. The intersection of the graphs are called the Feasible Region. The max and min of related function always occurs at a vertex (corner points) of the feasible range

Example 1:

A small company produces knitted a beanies and sweaters and sells them through a chain of specialty stores. The company is to supply the stores with a total of no more than 100 beanies and sweaters per day. The stores guarantee that they will sell at least 10 and no more than 60 beanies per day and at least 20 sweaters per day. The company makes a profit of $10 on each beanie and a profit of $12 on each sweater.

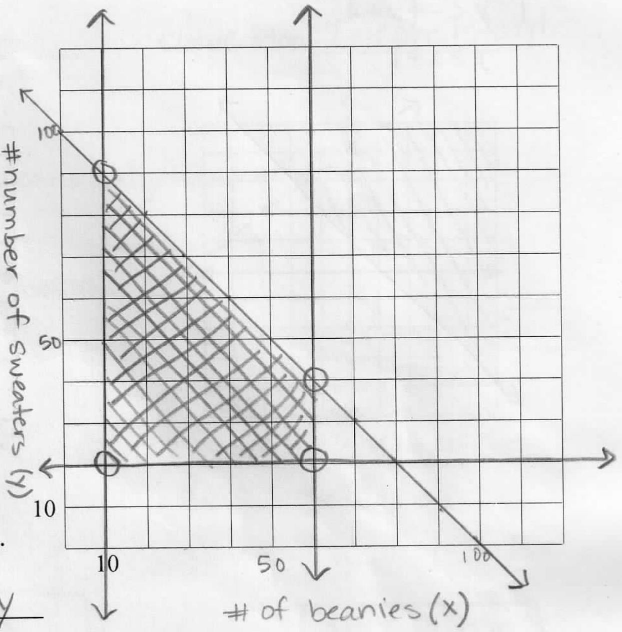
a) Write a system of inequalities to represent the information.
Let x = # of beanies
y = # of sweaters

$x + y \le 100 \rightarrow \boxed{y \le -x+100}$

$\boxed{10 \le x \le 60}$ $\boxed{x \ge 0}$

$\boxed{y \ge 20}$ $\boxed{y \ge 0}$

These inequalities are called Constraints.



b) Graph the inequalities.

Use x and y intercepts when possible.

The solution (**shaded region**) is called the Feasible region.

c) Write a function that represents the profit earned. $P(x, y) = 10x + 12y$

This is our Objective Function. An objective function can represent many things. The goal of **linear programming** is to maximize or minimize the objective function.
profit

d) The **Corner-Point Principle** states that the max or min values of the objective function will occur at one of the vertices of the feasible region. Identify and label the vertices in your graph. List them in the table below and calculate the profit at each one.

Vertex	Obj. Function: $P(x, y) = 10x + 12y$
(10, 20)	P(10, 20) = 10(10) + 12(20) = \$340
(10, 90)	P(10, 90) = 10(10) + 12(90) = \$1180
(60, 20)	P(60, 20) = 10(60) + 12(20) = \$1080
(60, 40)	P(60, 40) = 10(60) + 12(40) = \$840

Hint:
If you can't read the vertices from the graph, decide which two lines are intersecting and solve algebraically (use substitution or elimination.)

How many beanies and sweaters will give the maximum profit? 10 beanies + 90 sweaters

What is the maximum profit? $1180

Example 2:

A clothing factory produces casual shirts and dress shirts. No more than 1000 shirts can be made per hour and production costs may not exceed $8400 per hour. It costs $3 to make a casual shirt and $12 for a dress shirt. The profit is $0.85 per casual shirt and $3.00 per dress shirt.

Let x = # casual shirts

y = # dress shirts

	Casual shirts	Dress Shirts	Constraint
Amount	x +	y	≤ 1000
Cost	$3x$ +	$12y$	≤ 8400
Profit	$.85x$	$3y$	—

a) Write the constraints.

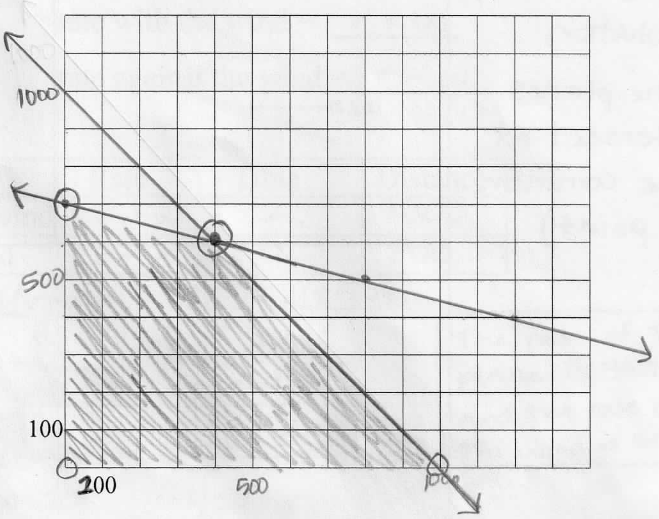
$\boxed{x \geq 0}$

$\boxed{y \geq 0}$

$x + y \leq 1000 \rightarrow \boxed{y \leq -x + 1000}$

$3x + 12y \leq 8400 \rightarrow \boxed{y \leq -\frac{1}{4}x + 700}$

b) Graph the feasible region.



c) Write the objective function. P(x, y) = $.85x + 3y$

Vertex	Obj. Function: P(x, y) = $.85x + 3y$
(0, 0)	P(0, 0) = .85(0) + 3(0) = $0
(0, 700)	P(0, 700) = .85(0) + 3(700) = $2100
(400, 600)	P(400, 600) = .85(400) + 3(600) = $2140
(1000, 0)	P(1000, 0) = .85(1000) + 3(0) = $850

d) How many dress and casual shirts need to be made to maximize the profit? What is the max profit?

The factory should make <u>400</u> casual shirts and <u>600</u> dress shirts for a maximum profit of <u>$2140</u>.

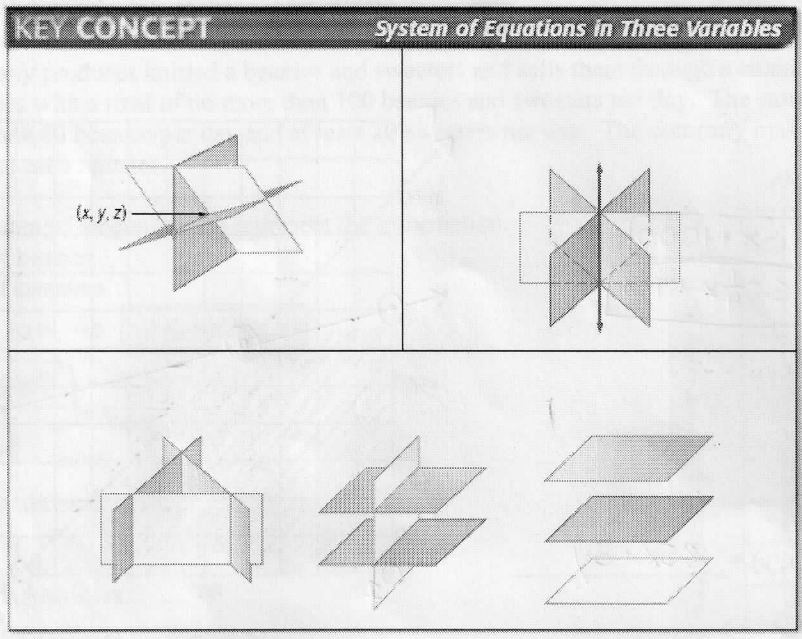
Section 3.5 – Solving Systems of Equations in Three Variables

Objective: To solve systems in 3 variables using substitution and elimination

Equations of Three Variables

Equations with 3 variables usually have the following variables: x, y, z. The graphical representation of a system in three variables to the Power of 1 is a plane. Below is graphical representation of the solutions of a system in three variables.

One Solution (The planes intersect at one common point)



Infinitely Many Solutions (The planes intersect at all the points along a common line)

No Solution (No point lies in all 3 planes)

Solving a System of Equation in Three Variables

Solving a system of equations in 3 variables is very similar to solving a system in 2-variables. Again there are 4 ways to solve a system of three variables: graph, subs, elimination, and matrices. We will focus on only two of these methods – substitution + elimination

Example: Solve the following system by substitution. (+ elimination)

$$\begin{cases} 3x+y-2z=22 & A \\ x+5y+z=4 & B \\ x=-3z & C \end{cases}$$

Ⓐ $3(-3z)+y-2z=22$

$-9z+y-2z=22$

$\boxed{-11z+y=22}$

Ⓑ $-3z+5y+z=4$

$\boxed{-2z+5y=4}$

$-5(-11z+y=22)$

$-2z+5y=4$

$55z-5y=-110$

$53z=-106$

$\frac{53z}{53}=\frac{-106}{53}$

$\boxed{z=-2}$

$x=-3(-2)$

$\boxed{x=6}$

$3(6)+y-2(-2)=22$

$18+y+4=22$

$22+y=22$

$-22 \quad -22$

$\boxed{y=0}$

$\boxed{(6,0,-2)}$

Examples: Solve the following systems by the elimination

1. $\begin{cases} x-2y+z=-1 \\ 2x+3y-2z=-3 \\ x+2y-2z=-13 \end{cases}$

2. $\begin{cases} 5x+3y+2z=2 \\ 2x+y-z=5 \\ x+4y+2z=16 \end{cases}$

on separate paper

1.) A, B, C

$$\begin{cases} x - 2y + z = -1 & (A) \\ 2x + 3y - 2z = -3 & (B) \\ x + 2y - 2z = -13 & (C) \end{cases}$$

AB

$2(x - 2y + z = -1)$

$2x + 3y - 2z = -3$

$2x - 4y + 2z = -2$

☆ $4x - y = -5$

BC

$(2x + 3y - 2z = -3)\,-1$

$x + 2y - 2z = -13$

$-2x - 3y + 2z = 3$

☆ $-x - y = -10$

☆☆

$4x - y = -5$

$(-x - y = -10)\,-1$

$x + y = 10$

$\frac{5x}{5} = \frac{5}{5}$

$\boxed{x = 1}$

☆ $4(1) - y = -5$

$4 - y = -5$

$-4 \qquad -4$

$-y = -9$

$\boxed{y = 9}$

A $(1) - 2(9) + z = -1$

$-17 + z = -1$

$+17 \qquad +17$

$\boxed{z = 16}$

$\boxed{(1, 9, 16)}$

2.) A, B, C

$$\begin{cases} 5x + 3y + 2z = 2 & (A) \\ 2x + y - z = 5 & (B) \\ x + 4y + 2z = 16 & (C) \end{cases}$$

AB

$5x + 3y + 2z = 2$

$2(2x + y - z = 5)$

$4x + 2y - 2z = 10$

☆ $9x + 5y = 12$

BC

$(2x + y - z = 5)\,2$

$x + 4y + 2z = 16$

$4x + 2y - 2z = 10$

☆ $5x + 6y = 26$

☆☆

$(9x + 5y = 12)\,6$

$(5x + 6y = 26)\,-5$

$54x + 30y = 72$

$-25x - 30y = -130$

$\frac{29x}{29} = \frac{-58}{29}$

$\boxed{x = -2}$

☆ $5(-2) + 6y = 26$

$-10 + 6y = 26$

$\frac{6y}{6} = \frac{36}{6}$

$\boxed{y = 6}$

$\boxed{(-2, 6, -3)}$

C $(-2) + 4(6) + 2z = 16$

$-2 + 24 + 2z = 16$

$22 + 2z = 16$

$-22 \qquad -22$

$\frac{2z}{2} = \frac{-6}{2}$

$\boxed{z = -3}$

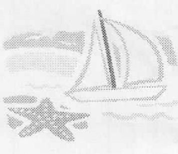
objective: To solve word problems involving distance, rate, time; mixtures, ~~and~~ using systems of equations

WORD PROBLEMS

Rate Word Problems. *Review*: **distance =** rate x time

Write a system for each problem and solve using substitution or elimination method.

1. A boat travels 30 km upstream (against current) in 3 hours. The boat travels the same distance downstream in 2 hours. Find the rate of the boat in still water and the rate of the current.



Let r = rate of boat

c = rate of current

rate downstream = r + c

rate upstream = r − c

	Rate	Time	Distance
downstream	b + c	2	30
upstream	b − c	3	30

$\frac{30}{2} = \frac{2(b+c)}{2}$

$15 = b + c$

$10 = b - c$

$\frac{25}{2} = \frac{2b}{2}$

$12.5 = b$

$\frac{30}{3} = \frac{3(b-c)}{3}$

$10 = b - c$

$10 = 12.5 - c$

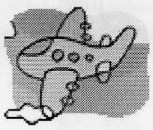
$-12.5 \quad -12.5$

$-2.5 = -c$

$2.5 = c$

The rate of the boat is 12.5 km/hr and the rate of the current is 2.5 km/hr

2. With the wind, a plane flew 1000 miles in 2 hours. Against the wind, the plane flew 800 miles in the same amount of time. Find the rate of the plane in still air and the rate of the wind.



Let r = rate of the plane

w = rate of the wind

rate with the wind = r + w

rate against the wind = r − w

	Rate	Time	Distance
downwind	r + w	2	1000
upwind	r − w	2	800

$\frac{1000}{2} = \frac{2(r+w)}{2}$

$500 = r + w$

$500 = 450 + w$

$w = 50$

$\frac{800}{2} = \frac{2(r-w)}{2}$

$400 = r - w$

$500 = r + w$

$\frac{900}{2} = \frac{2r}{2}$

$450 = r$

The rate of the plane is 450mph and the rate of the wind is 50mph

Mixture problems

Total Volume/Active Volume

1. A nurse wished to obtain 800 ml of a 7% solution of boric acid by mixing 4% and 12% solutions. How much of each should be used?

Let x = amount of 4%

y = amount of 12%

$x + y = 800 \rightarrow y = 800 - x$

$4x + 12y = 5600$

$4x + 12(800 - x) = 5600$

$4x + 9600 - 12x = 5600$

$-8x = -4000$

$x = 500$ mL

$y = 800 - 500 \rightarrow y = 300$ mL

		Amount	%	Total
4%	Solution 1	x	4	4x
12%	Solution 2	y	12	12y
	Mixture	800	7	5600

4% solution = 500 mL

12% solution = 300 mL

2. How much *pure* antifreeze must be added to 12 liters of a 40% solution of antifreeze to obtain a 60% solution?
Hint: Let x = # of liters of pure (100 %) antifreeze
y = # of liters of 60% solution

$x + 12 = y$

$100x + 480 = 60y$

$100x + 480 = 60(x + 12)$

$100x + 480 = 60x + 720$

$40x = 240$

$x = 6$

$6 + 12 = y$

$18 = y$

		Amount	%	Total
100%	Solution 1	x	100	100x
40%	Solution 2	12	40	480
60%	Mixture	y	60	60y

100% solution = 6L

60% solution = 18L

3. Penny wants to buy 40 pounds of a mixture of nuts by combining peanuts selling at $2.40 per pound with cashews selling at $3.20 per pound. If the mixture costs $2.70 per pound, how many pounds of each type of nut should she buy?

willing to pay

$-2.40(p + c = 40)$

$2.40p + 3.20c = 108$

$-2.40p - 2.40c = -96$

$.80c = 12$

$c = 15$ lbs

$p + 15 = 40$

$p = 25$ lbs

	Amount	Cost	Total
Peanuts	p	2.40	2.40p
Cashews	c	3.20	3.20c
Mixture	40	2.70	108

c = 15 lbs of cashews

p = 25 lbs of peanuts

Define the variables (be specific), write a system of equations, and solve. Write your final answer in a complete sentence.

4. The difference between two numbers is 16. Five times the smaller is the same as 8 less than twice the larger. Find the numbers.

Let x = small number

y = larger number

$y - x = 16 \rightarrow y = x + 16$

$5x = 2y - 8$

$5x = 2(x+16) - 8$

$5x = 2x + 32 - 8$

$5x = 2x + 24$

$\frac{3x}{3} = \frac{24}{3}$

$\boxed{x = 8}$

$y = 8 + 16$

$\boxed{y = 24}$

5. Nick has \$8 in nickels and dimes. All together he has 100 coins. How many of each type of coin does he have?

Let x = number of nickels

y = number of dimes

$x + y = 100 \rightarrow x = 100 - y$

$5x + 10y = 800$

$5(100 - y) + 10y = 800$

$500 - 5y + 10y = 800$

$5y = 300$

$y = 60$

$x = 100 - 60$

$x = 40$

$\boxed{\text{60 dimes, 40 nickels}}$

6. Tom and Jerry have saved \$4500 and want to invest it into 2 accounts. One account earns 5% annual interest and the other 9% annual interest. If they want to earn \$250 in interest per year, how much should they invest into each account?

Let x = amount of \$ in the 5% account

y = amount of \$ in the 9% account

* or eliminate

$X + Y = 4500$

$X = 4500 - Y$

$.05x + .09y = 250$

$.05(4500 - y) + .09y = 250$

$225 - .05y + .09y = 250$

$.04y = 25$

$y = \$625$

$x = 4500 - 625$

$x = \$3875$

\$625 into the account with 9% interest.
\$3875 into the account with 5% interest.