Date ____________ Notes 9.1 Mathematical Patterns

> Definition of a **sequence:** an ordered list of numbers, called terms.

You can describe sequences two different ways: Explicit formula or a Recursive formula.

Explicit Formula. Find the first four terms in the sequence. Then find a_{20}.

$a_n = 2n$

1st term: $a_1 = 2(1) = \boxed{2}$

2nd term: $a_2 = 2(2) = \boxed{4}$

3rd term: $a_3 = 2(3) = \boxed{6}$

4th term: $a_4 = 2(4) = \boxed{8}$

$a_{20} = 2(20) = \boxed{40}$

$a_n = -7n + 2$

1st term: $a_1 = -7(1) + 2 = \boxed{-5}$

2nd term: $a_2 = -7(2) + 2 = \boxed{-12}$

3rd term: $a_3 = -7(3) + 2 = \boxed{-19}$

4th term: $a_4 = -7(4) + 2 = \boxed{-26}$

$a_{20} = -7(20) + 2 = \boxed{-138}$

An Explicit Formula defines the nth term. Explain how you can find various terms.

- Replace n in the formula with the number of the term.
- Simplify to find the value.

Recursive Formula. Find the first five terms in the sequence.

$a_1 = 5$ ← 1st term!

$a_n = a_{n-1} + 3$

previous term!

$a_1 = \boxed{5}$

$a_2 = a_{2-1} + 3 = a_1 + 3 = 5 + 3 = \boxed{8}$

$a_3 = a_2 + 3 = \underline{8} + 3 = \boxed{11}$

$a_4 = a_3 + 3 = \underline{11} + 3 = \boxed{14}$

$a_5 = a_4 + 3 = 14 + 3 = \boxed{17}$

$a_1 = 3$ ← 1st term!

$a_n = 2a_{n-1} + 1$

$a_1 = 3$

$a_2 = 2 \cdot a_1 + 1 = 2 \cdot \underline{3} + 1 = \boxed{7}$

$a_3 = 2 \cdot a_2 + 1 = 2 \cdot \underline{7} + 1 = \boxed{15}$

$a_4 = 2 \cdot a_3 + 1 = 2 \cdot \underline{15} + 1 = \boxed{31}$

$a_5 = 2 \cdot a_4 + 1 = 2 \cdot \underline{31} + 1 = \boxed{63}$

Write a recursive formula for the sequence: 2, 6, 10, 14, ... *What's the pattern?

+4 +4 +4

two parts!

$a_1 = 2$

$a_n = a_{n-1} + 4$

A Recursive Formula defines the nth term by using the previous term.

Explain in your own words how you can find various terms.

- Plug in the previous term for a_{n-1}.
- Simplify to find the value.

Date ____________ Notes 9.2 Arithmetic Sequences

Suzie is saving her money up for a new bicycle. Her grandma gives her \$20 to get started, and then Suzie is going to put the same amount of money in her piggy bank each week. At the end of the second week, she has \$26. At the end of the 3rd week, she has \$32.

Fill in the table below to show the relationship between the number of weeks and the amount of money saved.

n (# of weeks)	1	2	3	4
a_n (money saved)	\$20	\$26	\$32	\$38

+6 +6 +6

This type of sequence is called Arithmetic because there is a common difference or an amount that is Added or Subtracted to find the next term.

$a_1 = 20$

$a_2 = 26 = 20 + 6 = 20 + 6(1)$

$a_3 = 32 = 20 + 12 = 20 + 6(2)$

$\vdots$

$a_n = 20 + 6(n-1)$

If the bike costs \$200, how long will she have to save?

$200 = 20 + 6(n-1)$

$30 = n - 1 \Rightarrow \boxed{n = 31 \text{ weeks}}$

Explicit Formula of an Arithmetic Sequence

In general, the terms in an Arithmetic Sequence will be of the form:

$$a_n = a_1 + (n-1)d$$

where $d =$ common difference $(a_2 - a_1)$

and $a_1 =$ 1st term

(*Note. recursive formula: $a_n = a_{n-1} + d$)

Are the following sequences arithmetic? If so, find the common difference, d.

a) 6, 10, 14, 18, . . .
+4 +4 +4
Yes! $d = 4$

b) 3, 6, 12, 24, . . .
+3 +6 +12
No!

c) $\pi, 2\pi, 3\pi, 4\pi, \ldots$
$+\pi$ $+\pi$ $+\pi$
yes! $d = \pi$

d) $\frac{17}{7}, \frac{12}{7}, 1, \frac{2}{7}, \ldots$
$+{-5/7}$ $-5/7$ $-5/7$
yes! $d = -5/7$

Find the explicit formula for the sequence. Then find a_{20}.

a) 2, –4, –10, –16, …
–6 –6 –6

$a_1 = 2 \quad d = -6$

$a_n = 2 + (n-1)(-6)$

$\boxed{a_n = -6n + 8}$

$a_{20} = -6(20) + 8 = \boxed{-112}$

b) $a_6 = 13$ and $a_9 = 22$

Method 1:

$13 = a_1 + (6-1)d \rightarrow 13 = a_1 + 5d$

$22 = a_1 + (9-1)d \rightarrow 22 = a_1 + 8d$

$9 = 3d$

$3 = d$

$a_1 = -2$

$a_n = -2 + (n-1)3$

$\boxed{a_n = 3n - 5}$

Method 2: 13 ___ ___ 22
a_6 a_7 a_8 a_9
d d d

$13 + 3d = 22$

$3d = 9$

$d = 3$

$13 = a_1 + (6-1)3$

$a_1 = -2$

$\boxed{a_n = 3n - 5}$

$\boxed{a_{20} = 55}$

Definition of an **Arithmetic Mean:**
Middle term(s) between 2 given terms in an Arithmetic Sequence

Example: find the arithmetic mean between 7.8 and 3.6.

$7.8 + 2d = 3.6 \qquad d = -2.1 \rightarrow 7.8 - 2.1 = \boxed{5.7}$

$2d = -4.2$

7.8, 5.7, 3.6
d d

Find *two* arithmetic means between 26 and 35.

$26 + 3d = 35$

26, 29, 32, 35

Date ______________ Notes 9.3 Geometric Sequences

A form of bacteria doubles every hour. How many bacteria cells will there be after 6 hours if you start with 5 cells? 160 bacteria

n (# of hours)	1	2	3	4	5	6
a_n (# of bacteria)	5	10	20	40	80	160

×2 ×2

This is an example of a Geometric Sequence because there is a Common ratio between successive terms. This means you multiply to get the next term. Do NOT DIVIDE.

Remember, in an arithmetic sequence there is a common difference which means you add or subtract to get the next term.

$a_1 = 5 = 5\cdot 1 = 5\cdot 2^0$

$a_2 = 10 = 5\cdot 2 = 5\cdot 2^1$

$a_3 = 20 = 5\cdot 4 = 5\cdot 2^2$

$a_4 = 40 = 5\cdot 8 = 5\cdot 2^3$

⋮

$a_n = a_1 \cdot r^{n-1}$

Explicit Formula of a Geometric Sequence

In general, the nth term of a Geometric Sequence is found by the formula:

$$a_n = a_1 \cdot r^{n-1}$$

where r = common ratio $\left(\frac{a_2}{a_1}\right)$

and a_1 = 1st term

(*Note. recursive formula: $a_n = a_{n-1}\cdot r$)

Determine whether the sequences are arithmetic, geometric or neither. If arithmetic, give d. If geometric, give r. -r gives an alternating sequence!

a) 6, 10, 14, 18, … +4 +4 +4 Arithmetic $d=4$

b) 2, 4, 8, 16, 32, … ×2 ×2 ×2 ×2 Geometric $r=2$

c) $5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots$ ·½ ½ ½ $r=\frac{1}{2}$

d) 3, -6, 12, -24, … -2 -2 -2 $r=-2$

Find the formula for a_n.
81, 54, 36, 24, …

$r = \frac{a_2}{a_1} = \frac{54}{81} = \frac{2}{3}$

$a_1 = 81$

$a_n = 81\cdot\left(\frac{2}{3}\right)^{n-1}$

If $a_2 = 6$ and $a_5 = 750$ in a geom. seq., find a_n and then a_7.

Method 1:

$6 = a_1(r)^{2-1} \rightarrow 6 = a_1\cdot r$

$750 = a_1(r)^{5-1} \quad 750 = a_1\cdot r^4$

$a_1 = \frac{6}{r} \rightarrow 750 = \frac{6}{r}\cdot r^4$

$750 = 6r^3$ $\quad a_1 = \frac{6}{5}$

$125 = r^3$

$r=5$

Method 2: $\frac{6}{a_2}\ \frac{\quad}{a_3}\ \frac{\quad}{a_4}\ \frac{750}{a_5}$ (r r r)

$6r^3 = 750$

$r^3 = 125$

$r=5$

$a_n = a_1(5)^{n-1}$

$6 = a_1(5)^{2-1}$

$a_1 = 6/5$

$\boxed{a_n = \frac{6}{5}(5)^{n-1}}$ $\quad a_7 = \frac{6}{5}(5)^6 = \boxed{18750}$

Definition of a **Geometric Mean**:
Middle term(s) between 2 given terms in a Geometric Sequence

Example: Find *two* geometric means between $\frac{1}{2}$ and $\frac{1}{54}$.

$\frac{2}{1}\cdot \frac{1}{2}r^3 = \frac{1}{54}\cdot\frac{2}{1}$

$r^3 = \frac{1}{27}$

$r = \frac{1}{3}$

$\frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$

$\frac{1}{6}\cdot\frac{1}{3} = \frac{1}{18}$

$\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}$ (r r r)

Date _________ Notes 9.4 Arithmetic Series

Definition of an **Arithmetic Series:** the SUM of the terms of an Arithmetic Sequence
A finite series has a 1st term (a_1) and a last term (a_n).
An infinite series continues without end.

Derivation: Given the sequence 1, 2, 3,….., 9, 10

Let $S_{10} = 1+2+3+4+5+6+7+8+9+10$

11

$= (1+10)+(2+9)+(3+8)+(4+7)+(5+6)$

$= 11+11+11+11+11$

$= 5(11)$ ← 1st + last

↖ # pairs

$= \boxed{55}$

In general, the (finite) sum of the n terms in an **arithmetic series** is found by the formula

$S_n = \frac{n}{2}(a_1 + a_n)$

n = # of terms

a_1 = 1st term

a_n = last term

**An infinite Arithmetic sum does not exist (or it* diverges*).*

Examples:

1. Use the formula to find
   $15+21+27+\ldots+63$ ($d=6$)

$a_1 = 15$ $\quad a_n = a_1 + (n-1)d$

$a_n = 63$ $\quad 63 = 15 + (n-1)6$

$n = ?$ $\quad 8 = n-1$

$n = 9$

$S_9 = \frac{9}{2}(15+63)$

$\boxed{S_9 = 351}$

Note: Summation Notation

"Sigma" $\sum_{n=3}^{5}(n^2) = 3^2 + 4^2 + 5^2 = \boxed{50}$

upper limit

"rule"

lower limit

2. Find the sum of the first 20 terms in the sequence: 8, 5, 2, … ($d = -3$)

$n = 20$

$a_1 = 8$ $\quad a_{20} = 8 + (19)(-3)$

$a_n = ?$ $\quad a_{20} = -49$

$S_{20} = \frac{20}{2}(8 + -49)$

$\boxed{S_{20} = -410}$

3. Use the formula to find $\sum_{j=1}^{16}(7-2j) = 5+3+1+\ldots+(-25)$

$n = 16$ $\quad a_1 = 7-2(1) = 5$

$a_1 = ?$ $\quad a_n = a_{16} = 7-2(16) = -25$

$a_n = ?$

$S_{16} = \frac{16}{2}(5+(-25))$

$\boxed{S_{16} = -160}$

4. What is the summation notation for the series $-5+2+9+16+\ldots+261+268$?

+7 +7

a) The first term is $a_1 = -5$

b) The common difference is $d =$ 7

c) The explicit formula is $a_n = -5+(n-1)7 = -12+7n$

d) The last term is 268

e) Use this value to find the number of terms (n): $268 = -12+7n$

$7n = 280$

$\therefore \sum_{k=1}^{40}(-12+7k)$

$= \frac{40}{2}(-5+268)$

$= 5260$

Date ________ Notes 9.5 Geometric Series

Definition of a **Geometric Series:** the SUM of the terms of a Geometric Sequence

Is the following a sequence or series? Is it arithmetic or geometric? Explain.

$$2 \underbrace{+}_{\times 3} 6 \underbrace{+}_{\times 3} 18 \underbrace{+}_{\times 3} 54 \underbrace{+}_{\times 3} 162$$

It is a Geometric Series.

Multiplying by the same number (3) each time, and terms are being added.

Find the sum. $\boxed{242}$

The (finite) sum of the first n terms of a **Geometric Series** is: $S_n = \dfrac{a_1(1-r^n)}{1-r}$ n = # terms a_1 = 1st term r = common ratio   *Note: Don't have to know the last term (a_n)!

Verify that the formula works for the example above:

$a_1 = 2$

$r = 3$

$n = 5$

$$S_n = \frac{2(1-3^5)}{1-3} = \frac{-484}{-2} = \boxed{242} \checkmark$$

1. Given: 400 + 300 + 225 +… Find S_{16}.

$a_1 = 400$

$r = \frac{300}{400} = \frac{3}{4}$

$n = 16$

$$S_{16} = \frac{400(1-(3/4)^{16})}{1-3/4}$$

$$= \boxed{1583.9638}$$

2. $\sum_{k=1}^{8} 4(-5)^{k-1} = 4 - 20 + \ldots - 312500$

$a_1 = 4(-5)^0 = 4$

$r = -5$

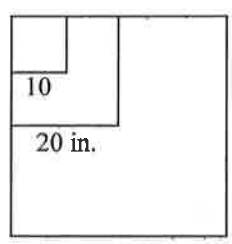
$n = 8$

$$S_8 = \frac{4(1-(-5)^8)}{1-(-5)}$$

$$= \frac{-1562496}{6}$$

$$\boxed{S_8 = -260416}$$

3. Smaller and smaller squares are formed as shown in the diagram.



40 in.

a) Find the sum of the perimeters of the first nine squares.
Hint: Is this a sequence or a series? Arithmetic or geometric?

Geometric Series: 4(40) + 4(20) + 4(10) + . . .

160 + 80 + 40 + . . .

$a_1 = 160$

$r = 1/2$

$n = 9$

$$S_9 = \frac{160(1-(1/2)^9)}{1-1/2}$$

$$\boxed{S_9 = 319.375 \text{ in}}$$

b) Find the sum of the areas of the first six squares.
Hint: Is this a sequence or a series? Arithmetic or geometric?

Geometric Series: $40^2 + 20^2 + 10^2 + \ldots$

1600 + 400 + 100 + . . .

$a_1 = 1600$

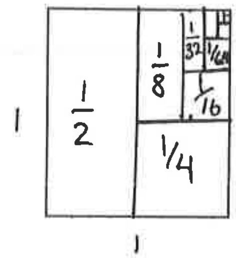
$r = 1/4$

$n = 6$

$$S_6 = \frac{1600(1-(1/4)^6)}{1-1/4}$$

Infinite Geometric Series

Example 1:



a) Suppose I had a square with side length = 1 as shown. What is the area?

1 un^2

b) Suppose I cut it in half. What is the area of the rectangle?

$\frac{1}{2} \text{ un}^2$

c) Suppose I keep cutting the remaining pieces in half and adding the area to the rectangle in #2.

What kind of sum have I created? Is the answer finite? If so, what is it?

Geometric Sum ($r = \frac{1}{2}$)

Infinite Sum - I can (theoretically) keep cutting the rectangles in half forever.

Sum = Area of Square = 1!

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$ is an example of an Infinite Geometric Series whose common ratio = $\frac{1}{2}$.

We say the series is Convergent. (sum exists!)

Example 2: Do you think the following infinite geometric sum exists? Why or why not? $1+2+4+8+16+\ldots$

No. The numbers keep getting bigger. Sum $\rightarrow \infty$

We say the series is Divergent. (Sum does not exist!)

> **Infinite Geometric Series**
>
> In general, if a geometric sequence has a common ratio r and $|r| < 1$, then the infinite geometric series is said to Convergent and the sum approaches a finite number. The sum is defined by the formula
>
> $$S_\infty = \frac{a_1}{1-r}$$

Check Ex 1:

$S_\infty = \frac{1/2}{1-1/2} = \frac{1}{2} \cdot \frac{2}{1} = 1 \checkmark$

If $|r| \geq 1$, then the Infinite geometric series diverges; the sum does NOT approach a finite number.

Determine if the following infinite geometric series converge or diverge. [Hint: find the common ratio, r.] If the series converges, find the sum.

$64+48+36+\ldots$

$r = \frac{48}{64} = \frac{3}{4} < 1 \checkmark$

converges!

$S_\infty = \frac{64}{1-3/4}$

$= \frac{64}{1} \cdot \frac{4}{1} =$

$\boxed{S_\infty = 256}$

$80-40+20-\ldots$

$r = \frac{-40}{80} = \frac{-1}{2} \rightarrow |\frac{-1}{2}| < 1 \checkmark$

Converges!

$S_\infty = \frac{80}{1-(-1/2)}$

$= \frac{80}{3/2}$

$= 80 \cdot \frac{2}{3}$ $\boxed{S_\infty = \frac{160}{3} = 58.\overline{3}}$

$8+9.6+11.52+13.824+\ldots$

$r = \frac{9.6}{8} = 1.2 > 1$ X

Diverges!

Repeating Decimals

Repeating Decimals can be written as fractions because they are just special cases of Infinite Geometric Series

Ex 1. $0.\overline{3} = 0.3 + 0.03 + 0.003 + 0.0003 + \ldots$

$a_1 = 0.3 \quad (3/10)$

$r = \frac{.03}{.3} = 0.1 \ (1/10)$

Thus, $S_\infty = \frac{0.3}{1-0.1} = \frac{0.3}{0.9} = \frac{3/10}{9/10} = \boxed{\frac{1}{3}}$

Ex. 2. $0.\overline{63} = 0.63 + 0.0063 + 0.000063 + \ldots$

$a_1 = 0.63 \quad (63/100)$

$r = \frac{0.0063}{0.63} = 0.01 \ (1/100)$

Thus, $S_\infty = \frac{0.63}{1-0.01} = \frac{63/100}{99/100} = \frac{63}{99} = \boxed{\frac{7}{11}}$

How is r determined?

It is what decimal place the number is being repeated to.
Ex #1 ($0.\overline{3}$) the 3 is in the 10th's place, Ex 2 ($.\overline{63}$) the last # is in the 100th's place.

Write as a fraction in simplest form without using your TI.

a) $0.\overline{2} = 0.2 + 0.02 + \ldots$

$a_1 = 2/10$

$r = 1/10$

$S_\infty = \frac{2/10}{1-1/10} = \frac{2/10}{9/10} = \boxed{\frac{2}{9}}$

b) $0.\overline{7} = 0.7 + 0.07 + \ldots$

$a_1 = 7/10$

$r = 1/10$

$S_\infty = \frac{7/10}{1-1/10} = \frac{7/10}{9/10} = \boxed{\frac{7}{9}}$

c) $0.\overline{54} = 0.54 + 0.0054 + \ldots$

$a_1 = 54/100$

$r = 1/100$

$S_\infty = \frac{54/100}{1-1/100} = \frac{54/100}{99/100}$

$= \frac{54}{99}$

$= \boxed{\frac{6}{11}}$

d) $0.\overline{123} = 0.123 + 0.000123 + \ldots$

$a_1 = 123/1000$

$r = 1/1000$

$S_\infty = \frac{123/1000}{1-1/1000} = \frac{123/1000}{999/1000}$

$= \frac{123}{999}$

$= \boxed{\frac{41}{333}}$

Key Words & Definitions:

Arithmetic: Adding/Subtracting same # each time (d)

Geometric: Multiplying by same # each time (r)

Sequence: A list of ordered terms (Ex: 1, 2, 3, ...)

Series: Sum of terms in a sequence (Ex: 1+2+3+...)

Explicit: Formula in terms of n. (Easy to find random terms)

Recursive: Formula in terms of previous term, a_{n-1}.

Finite: Has an end. (a_n is last term)

Infinite: Has no end; Goes on forever (∞)

Convergent: Sum/Limit exists and is equal to a (finite) number

Divergent: Sum/Limit does NOT exist. (Sum = ∞ or $-\infty$)